ABSTRACT

The arguments that lead to the $x^{1/3}$ dependence of wake width and the $x^{-2/3}$ dependence of axial velocity on axial distance in the far turbulent wake are reviewed. Assuming similarity of mean velocity profiles, the dominant terms in the Navier-Stokes equations for turbulent wake flow are used to derive these profiles with the help of Boussinesq's and Prandtl's hypotheses. The resulting profile has the same form as in the better known two-dimensional case. It is compared with an assumed Gaussian profile and both are compared with the Gaussian profile that results in the corresponding laminar case. The form of the eddy viscosity is also obtained in the two cases: for the former it is zero on the axis and rises to a maximum before falling to zero again on the wake boundary; whereas for a Gaussian profile it is constant across a cross section of the turbulent wake. In addition, the eddy viscosity has an $x^{-1/3}$ dependence in both cases, which is absent in the two-dimensional wake.

ON THE TURBULENT FAR-WAKE IN INCOMPRESSIBLE AXI-SYMMETRIC FLOW

by

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I. Introduction

The trails of axi-symmetric hypersonic blunt bodies flying through the atmosphere are at least partially turbulent. We expect that the turbulence has some effects, perhaps pronounced, on their radar and optical images, so that if we want to monitor such bodies, it becomes important for hypersonic flow theory to be able to predict these effects.

At the present, theory has very little to say about these turbulence effects. We have to consider the non-homogeneous shear flow of a compressible, chemically reacting gas. Statistical theories of turbulence exist only for homogeneous fluids, and even then there are few results for compressible fluids. On the other hand, the semi-empirical and heuristic procedures that have been developed for shear flows again, on the whole, avoid the compressible case. Nevertheless, since these semi-empirical procedures are all we have, it is the purpose of the present report to review them and to apply them specifically to the axi-symmetric wake. This is, in any case, a necessary first step.

Throughout this report then, we shall assume that we are dealing with a continuum with constant density and constant molecular viscosity. At every stage we shall compare the turbulent and the laminar cases. We begin with an order of magnitude analysis of the terms in the Navier-Stokes equations. Assuming the mean flow is steady and independent of the azimuthal angle in cylindrical coordinates it will be shown that the fluctuating part of the velocity,

$u_i = \bar{u}_i + u_i'$, enters the equations only through the combination $\overline{u_x' u_r'}$, where the bar denotes time averages and x and r refer to the axial and radial directions respectively. Next we present the well-known momentum theorem, which relates the total drag on the body to an integral of the axial velocity over a cross section of the wake, and of which extensive use is made subsequently. Some elementary considerations then lead to the conclusion that the width of the turbulent wake increases as $x^{1/3}$ and the velocity on the axis decreases as $x^{-2/3}$ (in the laminar case these dependencies are $x^{1/2}$ and x^{-1}). Finally, assuming similarity of velocity profiles in the far wake and making use of the concepts of an eddy viscosity and of Prandtl's mixing length, we derive the approximately Gaussian shape of the mean velocity profiles.

Recently a much more sophisticated analysis of the real hypersonic wake, taking compressibility into account, has been made by Lees and Hromas [5]. Lees and Hromas make extensive use of the concept of an eddy viscosity, of similarity-type agruments, and of various forms of the momentum theorem. Their principal innovation is to treat the more realistic picture of an inner turbulent core surrounded by an outer laminar wake, which in turn is surrounded by the shock shape and the undisturbed medium. As the core grows it eventually fills all of the wake, so that asymptotically their picture tends to the picture considered here. Their results agree well with the shadowgraph data of Slattery and Clay [6].

II. Order Of Magnitude Analysis of the Terms of the Navier-Stokes Equation

The Navier-Stokes equations and the continuity equation for steady flow with constant density and constant molecular viscosity in cylindrical coordinates (x, r, ϕ) are,

$$\frac{Du_x}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u_x \tag{1}$$

$$\frac{Du_r}{Dt} - \frac{u_\phi^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\phi}{\partial \phi}\right) \tag{2}$$

$$\frac{Du_\phi}{Dt} + \frac{u_r u_\phi}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \phi} + \nu\left(\nabla^2 u_\phi - \frac{u_\phi}{r^2} + \frac{2}{r^2}\frac{\partial u_r}{\partial \phi}\right) \tag{3}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} .$$

$$\frac{D}{Dt} \equiv u_x\frac{\partial}{\partial x} + u_r\frac{\partial}{\partial r} + \frac{1}{r}u_\phi\frac{\partial}{\partial \phi} + \left(\frac{\partial}{\partial t}\right)$$

and

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\phi}{\partial \phi} = 0 \tag{4}$$

For laminar axi-symmetric flow we put $u_\phi = 0$ and $\partial/\partial\phi = 0$. If we also replace u_x by $U + u_x$ where U is the constant incident velocity of the fluid in the rest frame of the body, then for such laminar flow (1) to (4) become,

$$\frac{Du_x}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u_x \tag{5}$$

$$\frac{Du_r}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\nabla^2 u_r - \frac{u_r}{r^2}\right) \tag{6}$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}$$

$$\frac{D}{Dt} \equiv (U + u_x)\frac{\partial}{\partial x} + u_r \frac{\partial}{\partial r}$$

and

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_r}{\partial r} + \frac{u_r}{r} = 0 \tag{7}$$

To get the corresponding equations for turbulent flow we write all flow variables as the sum of a time-averaged and turbulent part, $u_i = \bar{u}_i + u_i'$, etc. We again separate out the constant velocity at infinity by replacing $\bar{u}_x$ by $U + \bar{u}_x$ (note that this makes $\bar{u}_x$ negative). We assume that the mean flow is axi-symmetric, $\bar{u}_\phi = 0$. But it will not in general be true that $u_\phi' = 0$. Since the operations of averaging and differentiation commute, we can again put $\partial/\partial\phi = 0$. With these substitutions (1) to (4) become, after averaging,

$$(U + \bar{u}_x)\frac{\partial \bar{u}_x}{\partial x} + \bar{u}_r \frac{\partial \bar{u}_x}{\partial r} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} - \frac{\partial}{\partial x}\overline{u_x'^2} - \frac{1}{r}\frac{\partial}{\partial r}(r\,\overline{u_r' u_x'}) \tag{8}$$

$$\bar{u}_r \frac{\partial \bar{u}_r}{\partial r} + (U + \bar{u}_x)\frac{\partial \bar{u}_r}{\partial x} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial r} - \frac{1}{r}\frac{\partial}{\partial r}(r\,\overline{u_r'^2}) - \frac{\partial}{\partial x}(\overline{u_r' u_x'}) + \frac{\overline{u_\phi'^2}}{r} \tag{9}$$

$$0 = \frac{\partial}{\partial r}(\overline{u_\phi' u_r'}) + \frac{\partial}{\partial x}\overline{u_\phi' u_x'} + \frac{2}{r}\overline{u_\phi' u_r'} \tag{10}$$

and

$$\frac{\partial \bar{u}_x}{\partial x} + \frac{\partial \bar{u}_r}{\partial r} + \frac{\bar{u}_r}{r} = 0 \tag{11}$$

where an important simplification has been effected by dropping all terms involving the molecular viscosity in (1), (2), and (3). This is the first of many assumptions arising from empirical knowledge. The terms involving mean values of the fluctuating velocity components in (8), (9), and (10), the "turbulent stresses," are known to be several orders of magnitude greater than the molecular viscous stresses. The data of Townsend [4], for instance, on the flow behind an infinite cylinder show that the mean properties of the wake are independent of Reynolds number for Reynolds numbers $R = \frac{Ud}{\nu}$ greater than about 800, d being the diameter of the cylinder. This justifies the neglect of the molecular viscous stresses. On the other hand, we also know that molecular viscosity is the only available mechanism for the final decay of the turbulence. Any possible exact treatment of equations (8) to (11) would therefore be incapable of describing the process of dissipation. In fact, of course, equations (8) to (11) cannot be solved without making some semiempirical assumptions about the form of the turbulent stresses, and these assumptions will implicitly contain information on the effects of the molecular viscous stresses as well.

The analogy between the molecular viscous stresses in equations (1) to (3) and the turbulent stresses in equations (8) to (10) is most clearly seen by writing the Navier-Stokes equations compactly in rectangular coordinates. Using the summation convention and letting i = 1, 2, 3 refer to x, y, z coordinates, the laminar equations are

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{1}{\rho}\frac{\partial}{\partial x_j}(\tau_\ell)_{ij}$$

where

$$(\tau_\ell)_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

and the turbulent equations without terms involving the molecular viscosity are

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_j} + \frac{1}{\rho}\frac{\partial}{\partial x_j}(\tau_t)_{ij}$$

where

$$(\tau_t)_{ij} = -\rho\, \overline{u_i' u_j'} \ .$$

The laminar equations form a closed system: three equations for three unknowns. The turbulent equations involve in addition to $\bar{u}_i$ the six independent components of the turbulent stress tensor $(\tau_t)_{ij}$. If we hope to solve the latter we must make some assumptions relating $(\tau_t)_{ij}$ to the mean velocity or its gradient.

To effect an order of magnitude analysis of equations (8) to (11) we follow Hinze [2] and introduce longitudinal and transverse length scales, L_x and L_r, which are distances characteristic of significant changes in the mean flow. For wake flow, expecially at high velocities, we assume

$$L_r \ll L_x \ . \qquad (12)$$

If U_x and U_r are the corresponding longitudinal and transverse velocity scales (U_x gives the scale for $\bar{u}_x$ only, i.e. it does not include the velocity at infinity U), then it follows from the continuity equation (11) that

$$\frac{U_r}{L_r} \sim \frac{U_x}{L_x} \ ,$$

so that

$$U_r \ll U_x . \tag{13}$$

Let v^2 be the scale for $\overline{u_r'^2}$, $\overline{u_x'^2}$, $\overline{u_\phi'^2}$, which are all assumed to be of the same order of magnitude (c.f. Townsend [4] for the slightly different case of an infinite cylinder). If correlation coefficients R_{ij} are defined by

$$R_{ij} \equiv \frac{\overline{u_i' u_j'}}{\left[\overline{u_i'^2}\ \overline{u_j'^2}\right]^{1/2}}$$

then the scale size of $\overline{u_i' u_j'}$ in general will be $R_{ij} v^2$.

With these scale sizes the orders of magnitude of the terms in (8) are (after multiplying through by L_x/U_x^2),

$$\frac{U}{U_x} + 1 + 1 \sim -\frac{(\Delta p)_x}{\rho U_x^2} - \frac{v^2}{U_x^2} - R_{rx}\frac{v^2}{U_x^2}\frac{L_x}{L_r} \tag{14}$$

and those in (9) are (after multiplying through by L_x/U_rU_x)

$$1 + \frac{U}{U_x} + 1 \sim -\frac{(\Delta p)_r}{\rho U_x^2}\frac{L_x^2}{L_r^2} - \frac{v^2}{U_x^2}\frac{L_x^2}{L_r^2} - R_{rx}\frac{v^2}{U_x^2}\frac{L_x}{L_r}, \tag{15}$$

where $(\Delta p)_x$ and $(\Delta p)_r$ are the gross longitudinal and lateral pressure differences.

We now make the far wake assumption, which is that the difference in axial velocity on the axis and very far from the axis is small, i.e.

$$U_x \ll U . \tag{16}$$

The data of Townsend [4] , again for an infinite cylinder, show that this leads to no contradiction for axial distances downstream from the body greater than about 100 body diameters.

If we make the further reasonable assumptions that $R_{rx} \sim O(1)$, and that v^2/U_x^2 is at most $O(1)$, then it follows from (14) and (12) that U/U_x is at most of order L_x/L_r. Therefore, in (15), L.H.S. $\ll$ R.H.S., and therefore, the pressure term on the R.H.S. of (15) must balance the first turbulence term (which represents the second and fourth terms on the R.H.S. of (9)). Thus (9) becomes

$$\frac{1}{\rho}\frac{\partial \bar{p}}{\partial r} + \frac{1}{r}\frac{\partial}{\partial r}(r\,\overline{u'^2_r}) - \frac{\overline{u'^2_\phi}}{r} = 0,$$

which upon integration gives

$$\bar{p}/\rho + \overline{u'^2_r} + \int \frac{1}{r}\left[\overline{u'^2_r} - \overline{u'^2_\phi}\right] dr = \text{const.} \tag{17}$$

Differentiating now with respect to x gives

$$\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x} + \frac{\partial}{\partial x}\overline{u'^2_r} + \frac{\partial}{\partial x}\int \frac{1}{r}\left[\overline{u'^2_r} - \overline{u'^2_\phi}\right] dr = 0,$$

which shows that the pressure term in (14) is of order v^2/U_x^2, and may therefore be neglected in comparison with the second turbulence terms. (14) then shows that (8) becomes (after dividing through by U^2),

$$\frac{\partial}{\partial x}\frac{\bar{u}_x}{U} = \frac{1}{r}\frac{\partial}{\partial r}\left(-r\frac{\overline{u'_r u'_x}}{U^2}\right). \tag{18}$$

This is the equation that will be used in the sequel to derive the mean properties of the turbulent wake.

Equation (17) would give information on the mean pressure distribution if suitable assumptions about $\overline{u_i'^2}$ could be introduced. We also have not yet considered equation (10); the order of magnitude analysis of (10) shows that

$$\frac{R_{\phi r}}{R_{\phi x}} \sim \frac{L_r}{L_x} \ll 1$$

i.e., the axial component of the turbulent velocity is much more strongly correlated with the azimuthal component than is the radial component. This seems to suggest that eddies with their axes parallel to the main stream direction are less prevalent than eddies with their axes pointing in a radial direction.

If we make a similar analysis of the equations for the laminar wake, (5), (6), (7), then instead of (18) we get

$$\frac{\partial}{\partial x} \frac{u_x}{U} = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\nu}{U} \frac{\partial}{\partial r} \frac{u_x}{U}\right) . \qquad (19)$$

III. The Momentum Theorem

In this section we closely follow the treatment of Landau and Lifshitz [1] and write the Navier-Stokes (laminar) equations (1), (2), (3) in rectangular coordinates in the form

$$\frac{\partial}{\partial t} (\rho u_i) = - \frac{\partial P_{ik}}{\partial x_k}$$

where P_{ik} is the momentum flux density tensor,

$$P_{ik} = p\,\delta_{ik} + \rho u_i u_k - \mu\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right) . \tag{20}$$

Now the total momentum transported by the fluid through any closed surface surrounding the body is equal to the force on the body, F_i,

$$F_i = \oint P_{ik}\, dA_k \quad . \tag{21}$$

Let the closed surface be a cylinder with axis along the flow axis, its curved surface infinitely far away from the axis, one end upstream of the body (which we may picture as a sphere), and the other end across the wake. As before, replace u_i by $U_i + u_i$, where $U_i = (U, o, o)$, the x direction being the flow direction, and replace p by P + p, where P is the pressure of the undisturbed stream. Making use of the following,

a) $\oint$ (const.) dA over any closed surface $= 0$
b) $\oint \rho u_i\, dA_i$ = mass flux through closed surface $= 0$
c) $|u_x| \ll U$

(21) becomes,

$$F_i = \oint \left[p\,\delta_{ik} + \rho\, U_k u_i - \mu\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)\right] dA_k \ .$$

The parts of the integral over the curved surface of the cylinder and over the upstream end are zero because p = u = 0 there. The drag (i.e. F_x) therefore is

$$F = -\iint \left(p + \rho U\, u_x - 2\mu \frac{\partial u_x}{\partial x}\right) dA \tag{22}$$

where the integral is over a cross section of the viscous wake. But inside the wake $p \sim \rho\, {u'}^2 \ll \rho U u_x$, and the viscous term is also small, expecially at high Reynolds numbers (we are still considering a laminar wake, however). Thus the drag becomes

$$F_\ell = -\rho U \iint u_x \, dA. \tag{23}$$

If the wake is turbulent then all velocities in (20) must be replaced by mean velocities, and the molecular viscous term in (20) must be replaced by $+\overline{u_i' u_k'}$. The analysis then goes through as for the laminar wake, and (22) becomes

$$F_t = -\iint (\bar{p} + \rho U \bar{u}_x + 2\,\overline{{u_x'}^2})\, dA .$$

(17) shows that $\rho U \bar{u}_x$ is again the dominant term, so that

$$F_t = -\rho U \iint \overline{u_x} \, dA. \tag{24}$$

In the next section we shall make immediate use of (23) and (24).

IV. The Width of the Wake and the Velocity on the Axis

A. Laminar Wake

Let $b(x)$ be the width of the wake, i.e. assume that at any fixed x u_x becomes negligible for $r \geq b(x)$. Then in the equation for the laminar wake, (19), the order of magnitude of the L.H.S. is $u_x/(x-x_o)\, U$, where x_o is some virtual origin of the wake (close to the body). The order of magnitude of the R.H.S. of (19) is $\nu u_x / b^2 U^2$. Equating the two gives,

$$b_\ell(x) \sim \left(\frac{\nu}{U}\right)^{1/2} (x-x_o)^{1/2} \tag{25}$$

Further, from the momentum theorem (23) we have

$$F_\ell \sim -\rho U u_{xo} b^2$$

where u_{xo} is the velocity on the axis, therefore

$$\frac{u_{xo}}{U} \sim -\frac{F_\ell}{\rho\nu U}(x-x_o)^{-1}. \tag{26}$$

B. Turbulent Wake

Before any information about the turbulent wake can be extracted from equation (18) some assumption relating the turbulent shear stress $-\rho\overline{u_r'u_x'}$ to the mean velocity must be made. The assumption that is usually made is implied in the terminology that we have already been using: in analogy with the molecular viscous stresses we put,

$$(\tau_t)_{xr} \equiv -\rho\overline{u_x'u_r'} = \rho\epsilon\frac{\partial\bar{u}}{\partial r}. \tag{27}$$

This is Boussinesq's hypothesis. ϵ is called the dynamic eddy (or turbulent) viscosity. In contrast to the laminar case we cannot, in general, go on to assume that ϵ is an intrinsic property of the fluid, even for an incompressible fluid. To relate the eddy viscosity in turn to the mean properties of the flow the usual procedure is to use Prandtl's mixing length hypothesis. In analogy to the mean free path of gas molecules this introduces the concept of a length ℓ, characteristic of the decay of fluid "particles" (i.e. eddies).*

* A good discussion of Prandtl's mixing length hypothesis is given by Schlichting (3), p. 477.

We therefore assume,

$$\epsilon = \ell^2 \left|\frac{\partial \bar{u}}{\partial r}\right| , \tag{28}$$

and we make the further reasonable assumption that the mixing length, ℓ, is of the same order as the width, b_t, of the turbulent wake,

$$\ell \sim b_t . \tag{29}$$

With the three assumptions (27), (28), (29) the order of magnitude of the R.H.S. of equation (18) becomes

$$\frac{\overline{u'_r u'_x}}{b_t U^2} \sim \frac{\epsilon \, \bar{u}_{xo}}{b_t^2 U^2} \sim \frac{\bar{u}_{xo}^2}{b_t U^2} ,$$

and equating this to the order of magnitude of the L.H.S., $\bar{u}_x/(x-x_o)\,U$, gives

$$\frac{\bar{u}_{xo}}{U} \sim \frac{b_t(x)}{(x-x_o)} . \tag{30}$$

From the momentum theorem (24) we have

$$F_t \sim -\rho U \, \bar{u}_{xo} \, b_t^2(x) . \tag{31}$$

Therefore (30) and (31) finally give

$$b_t(x) \sim \left(\frac{F_t}{\rho U^2}\right)^{1/3} (x-x_o)^{1/3} \tag{32}$$

and

$$\frac{\bar{u}_{xo}}{U} \sim - \left(\frac{F_t}{\rho U^2}\right)^{1/3} (x-x_o)^{-2/3} \quad . \tag{33}$$

Comparing (32) and (33) with (25) and (26) notice that the laminar wake widens more rapidly and its mean axial velocity falls off more rapidly than the turbulent wake. This may seem paradoxical, since one would expect that turbulent dissipation, which is so much greater than molecular viscous dissipation, would produce a shorter turbulent wake. Actually, in order to define the "length" of a wake one would have to introduce some cut-off criterion, such as measuring the wake only to the point at which the velocity on the axis has reached 99% of the incident stream velocity. Hence the relative magnitudes of the coefficients of the powers of x above are important. Unfortunately the theory does not give any clues as to these magnitudes.

It should be mentioned that Landam and Lifshitz [1] , p. 136, derive (32) and (33) by different reasoning without invoking either Boussinesq's or Prandtl's hypothesis. Yet another approach is that outlined below.

V. Mean Velocity Distribution in the Wake

If we assume that the mean flow pattern in the far wake has self-preserving form we can determine what that form is. Here we follow the procedure of Hinze [2] , who treated the slightly different case of a two dimensional wake.

Make a transformation of equation (18) from the independent variables (x, r) to the new independent variables (ξ, η), where

$$\xi = x, \quad \text{and} \quad \eta = \frac{r}{b_t(x)} \tag{34}$$

and put

$$\frac{\bar{u}_{xo}}{U} = \psi(\xi), \quad \frac{\bar{u}_x}{\bar{u}_{xo}} = f(\eta), \quad -\frac{\overline{u'_r u'_x}}{\bar{u}_{xo}^2} = h(\eta) \tag{35}$$

Thus (18) becomes

$$f \frac{d\psi}{d\xi} - \frac{1}{b_t} \eta \psi \frac{db_t}{d\xi} \frac{df}{d\eta} = \frac{1}{\eta b_t} \psi^2 \frac{d}{d\eta} (\eta h) \tag{36}$$

With the same substitutions the momentum theorem (24) gives

$$F_t = -2\pi \rho U^2 b_t^2 \psi \int_0^\infty f \eta \, d\eta$$

or

$$b_t^2 \psi = \text{const.} = -A \tag{37}$$

where

$$A = \frac{F_t}{2\pi\rho U^2} \left[\int_0^1 f \eta \, d\eta \right]^{-1} \tag{38}$$

The integral in the momentum theorem is over an infinite plane cross section of the wake, but since by (35) the integrand is essentially negligible outside the wake $r > b_t(x)$, the integration can be taken from 0 to 1 as in (38).

Eliminate ψ from (37) and (36) so that (36) may be written in the form

$$b_t^2 \frac{db_t}{d\xi} = A \left[\frac{d}{d\eta} (\eta h)\right] \left[\frac{d}{d\eta} (\eta^2 f)\right]^{-1} \tag{39}$$

Since the L.H.S. of (39) is a function of ξ only, and the R.H.S. is a function of η only, both sides must equal a constant. Let

$$b_t^{\,2}\frac{db_t}{d\xi} = \text{const.} = K$$

so that

$$b_t(x) = \left[3K\,(x-x_o)\right]^{1/3}, \tag{32a}$$

and we have recovered (32) without the use of either Boussinesq's or Prandtl's hypothesis.

From (39) we also have

$$K\,\eta\,f = A\,h\,. \tag{40}$$

With Boussinesq's hypothesis (27) we now get

$$h = -\frac{\epsilon}{AU}\,b_t\,\frac{df}{d\eta} \tag{41}$$

and Prandtl's mixing length hypothesis (28) together with an assumption of the same form as (29),

$$\ell = \beta b_t\,, \tag{42}$$

where β is a dimensionless constant of order unity, gives

$$\epsilon = \beta^2 AU\frac{1}{b_t}\,\frac{df}{d\eta}\,. \tag{43}$$

Substituting (41), (42), (43) in (40) gives

$$K\,\eta\,f = -\,\beta^2 A\,\frac{df}{d\eta}\cdot\left|\frac{df}{d\eta}\right| \tag{44}$$

of which the solution with $f = 1$ at $\eta = 0$ is

$$f(\eta) = \left[1 - \left(\frac{K}{9\,\beta^2 A}\right)^{1/2} \eta^{3/2}\right]^2 \qquad (45)$$

The constants A and K have the dimensions of $(\text{length})^2$. Since b_t is the width of the wake, i.e. $\bar{u}_x \approx 0$ for $r \geq b_t$, we must have $f = 0$ when $\eta = 1$. Hence the constant in (45) is given by

$$\left(\frac{K}{9\,\beta^2 A}\right) = 1 \qquad (46)$$

and (45) becomes

$$f(\eta) = \left[1 - \eta^{3/2}\right]^2 \qquad (47)$$

Furthermore, using (47) in (38) gives, with (47),

$$A = \frac{70}{9}\,\frac{F_t}{2\pi\rho U^2}\,, \quad K = 70\,\beta^2\,\frac{F_t}{2\pi\rho U^2} \qquad (48)$$

so that β remains as a single undetermined constant. The previous result (33) can be recovered from (35), (37), and (48),

$$\frac{\bar{u}_{xo}}{U} = -\frac{1}{9}\left(\frac{70}{2\pi\beta}\right)^{1/3}\left(\frac{F_t}{\rho U^2}\right)^{1/3}(x-x_o)^{-2/3}$$

$$= -0.248\left(\frac{F_t}{\beta\rho U^2}\right)^{1/3}(x-x_o)^{-2/3} \qquad (33a)$$

and we also have from (35(, (32a) and (47),

$$\frac{\bar{u}_x}{U} = \frac{\bar{u}_{xo}}{U}\left[1 - \left(\frac{2\pi\, U^2}{210\, \beta^2 F_t}\right)^{1/2} \left(\frac{r^3}{x-x_o}\right)^{1/2}\right]^2 \tag{49}$$

and from (43), (32a), and (48),

$$\frac{\epsilon}{U} = 3\, \beta^2 A \left[3K\,(x-x_o)\right]^{-1/3} \eta^{1/2} \,(1-\eta^{3/2}) \tag{50}$$

Thus according to (49) and (50) $\bar{u}_x$ has a shape at any cross section of the wake which is close to Gaussian, but with a well-defined boundary ($\eta = 1$), whereas the eddy viscosity ϵ is zero on the axis and on the boundary and has a maximum at $\eta = 1/3$. In contrast to the two-dimensional case treated by Hinze [2] (and most others) ϵ also depends on x, falling off as $x^{-1/3}$.

To compare the turbulent wake with a laminar one consider equation (19) for the laminar wake. Without further hypotheses it is easy to verify that

$$\frac{u_x}{U} = \frac{N}{(x-x_o)} \exp\left[-\frac{Ur^2}{4\,\nu(x-x_o)}\right] \tag{51}$$

is a solution. The momentum theorem (23) again serves to determine the constant of integration N,

$$N = \frac{F_\ell}{4\pi\rho U\nu} \tag{52}$$

Just as for the turbulent case we can again recover the previous order of magnitude results (25) and (26) from (51), i.e.

$$b_{\ell}(x) = \left(\frac{4\nu}{U}\right)^{1/2} (x-x_o)^{1/2} \tag{25a}$$

and

$$\frac{u_{xo}}{U} = -\frac{F_{\ell}}{4\pi\rho U\nu} (x-x_o)^{-1} \tag{26a}$$

In this case, however, since an exponential form for u_x does not yield a well-defined boundary, b(x) has to be redefined: in (25a) we have implicitly defined b(x) as the radius of the wake at $\frac{1}{e}$ of the central maximum. (51) for the laminar wake can then be compared with (49) for the turbulent wake.

Does the non-Gaussian shape (49) of the turbulent wake have any real significance? It is certainly true that real turbulence cannot always extend indefinitely in a transverse direction as a Gaussian shape would imply, i.e. at sufficient distance from the axis the flow will at times definitely be laminar. On the other hand, there is no difference from the experimental point of view between fluctuations of zero amplitude and fluctuations of an amplitude too small to be detected. An alternative procedure would therefore be to assumed boldly from the start that the turbulent mean velocity has a Gaussian shape. Townsend's data for the two-dimensional wake certainly justify this procedure. This inductive approach is essentially that due to Reichardt (see Hinze [2], Ch. 5 and 6).

Thus we assume similarity of velocity profiles as before, (34) and (35), and therefore derive (32a) and (40) again. We then put

$$\frac{\bar{u}_x}{\bar{u}_{xo}} = f(\eta) = \exp\left[-\eta^2\right] \tag{53}$$

where b_t in (34) now means that half-width at $\frac{1}{e}$ of the central maximum as in the laminar case. Instead of (48) we now obtain

$$A = \frac{2e}{e-1} \frac{F_t}{2\pi\rho U^2} = 3.2 \frac{F_t}{2\pi\rho U^2} \qquad (54)$$

and K remains as a disposable constant (together with x_o, the virtual origin of the wake).

We can now determine what (53) implies about the eddy viscosity (for which we have no real need, however). Substituting (53) into (40), and then using (41) to eliminate $h(\eta)$ gives

$$\frac{\epsilon}{U} = \left(\frac{K^2}{24}\right)^{1/3} (x-x_o)^{-1/3} \qquad (55)$$

Thus in this model the eddy viscosity is constant across a cross section of the wake, which fits the experimental data much better than expression (50). We still have the $x^{-1/3}$ dependence of ϵ.

Finally we may remark that the agreement with experiment of the profile (53) can be considerably improved by introducing the empirical concept of an intermittency factor, Ω, the ratio between the time during which turbulence occurs at a given point and the total characteristic flow time. Ω takes account of the fluctuating nature of the turbulent wake boundary (which in a real case may even momentarily touch the axis). Ω itself has an approximately Gaussian shape. Its effect is to flatten the inner part of the velocity profile (53) and to steepen the outer part (see Hinze [2], Ch. 6).

VI. References

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